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Measure Properties of the Set of Initial Data yielding Non Uniqueness for a class of Differential Inclusions

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Abstract

We study uniqueness property for the Cauchy problem $x' \in \partial V(x)$, $x(0) = \xi$, where $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a locally Lipschitz continuous, quasiconvex function (i.e. the sublevel sets $\{V \leq c\}$ are convex) and $\partial V(x)$ is the generalized gradient of V at x . We prove that if $0 \notin \partial V(x)$ for $V(x) \geq b$, then the set of initial data $\xi \in \{V = b\}$ yielding non uniqueness of solution in a geometric sense has $(n-1)$ -dimensional Hausdorff measure zero in $\{V = b\}$.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

In this work we study uniqueness property for Cauchy problems of the type

$$(\mathcal{P}_\xi) \quad \begin{cases} x'(t) \in \partial V(x(t)) \\ x(0) = \xi. \end{cases}$$

When $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and locally Lipschitz continuous, $\partial V(x)$ is the sub-differential in the sense of convex analysis and the multifunction $x \mapsto \partial V(x)$ is a maximal monotone operator in \mathbb{R}^n . This implies that for every $\xi \in \mathbb{R}^n$ the problem (\mathcal{P}_ξ) admits at least a solution, which is unique in the past. Moreover, as proved by Cellina [C] in a more general setting, for almost every $\xi \in \mathbb{R}^n$ the problem (\mathcal{P}_ξ) admits a unique solution.

The interest in this subject arises from some questions in the calculus of variations. In this framework V would be a solution to a minimization problem and the hypothesis of convexity for V seems to be too strong.

It is more natural to consider the case in which $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz continuous and its sublevel sets are convex. In this case $\partial V(x)$ is the generalized gradient, introduced by Clarke [Cl]. Since the multifunction $x \mapsto \partial V(x)$ is upper semicontinuous with compact convex values, for any $\xi \in \mathbb{R}^n$ the problem (\mathcal{P}_ξ) admits at least a local solution.

We point out that in general the operator $\partial V(\cdot)$ is not monotone (actually it is monotone if and only if V is convex, see [Cl, Proposition 2.2.9]) and therefore nothing can be said about uniqueness of solutions of (\mathcal{P}_ξ) . Hence it is natural to consider uniqueness property in a geometrical sense. Precisely, two solutions to (\mathcal{P}_ξ) are geometrically distinct in the future if, for positive times they draw two different curves in \mathbb{R}^n .

We prove the following result.

Theorem. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and $a, b \in \mathbb{R}$, $b > a$, satisfy:*

- (V₁) V is locally Lipschitz continuous;*
- (V₂) the sets $\{V \leq c\}$ are convex for every $c \geq a$;*
- (V₃) $0 \notin \partial V(x)$ for every $x \in \{V \geq b\}$.*

Let Ω be the set of the points $\xi \in \mathbb{R}^n$ such that the problem (\mathcal{P}_ξ) admits (at least) two geometrically distinct solutions in the future. Then the $(n-1)$ -dimensional Hausdorff measure of the set $\{V = b\} \cap \Omega$ is zero.

As immediate consequence of the theorem we get the following result.

Corollary. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function whose sublevel sets are convex and $0 \notin \partial V(x)$ for $V(x) > \inf V$. Then the n -dimensional Lebesgue measure of Ω is zero.*

Remarks. (i) We point out that, thanks to (V₁)–(V₃), for any $c \geq b$ the level set $\{V = c\}$ is covered by a countable collection of $(n-1)$ -dimensional rectifiable sets.

(ii) When V is convex, the generalized gradient $\partial V(x)$ coincides with the standard subdifferential in the sense of convex analysis. In addition two different solutions to (\mathcal{P}_ξ) are geometrically distinct in the future.

(iii) The hypothesis (V_3) cannot be omitted, as we emphasize in the example below.

2. NOTATIONS AND PRELIMINARY RESULTS

Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function. By Rademacher's Theorem, the function V is differentiable almost everywhere (in the sense of Lebesgue measure) on \mathbb{R}^n . Moreover the generalized gradient $\partial V(x)$ can be defined as the set

$$(2.1) \quad \partial V(x) = \overline{\text{co}} \{ \lim \nabla V(x_i) : (x_i) \subset D, \lim x_i = x, \exists \lim \nabla V(x_i) \}$$

where $D = \{ x \in \mathbb{R}^n : \exists \nabla V(x) \}$.

A first useful property that we need concerns the relationship between the generalized gradient $\partial V(x)$ and the cone of normals to $\{V \leq V(x)\}$ at the point x .

To state it, we introduce some standard definitions, following [AC] and [Cl]. Given a convex closed nonempty subset C of \mathbb{R}^n and a point $x \in \mathbb{R}^n$ we set $d_C(x) = \inf \{ |x - y| : y \in C \}$. It holds that:

(2.2) the function $x \mapsto d_C(x)$ is convex and $|d_C(x) - d_C(y)| \leq |x - y|$ for any $x, y \in \mathbb{R}^n$;

(2.3) for any $x, v \in \mathbb{R}^n$ there exists $\lim_{t \rightarrow 0+} \frac{1}{t} (d_C(x + tv) - d_C(x)) = d'_C(x; v)$.

Then we define the tangent cone to C at x as

$$(2.4) \quad T_C(x) = \{ v \in \mathbb{R}^n : d'_C(x; v) = 0 \}$$

and the normal cone to C at x as

$$(2.5) \quad N_C(x) = \{ y \in \mathbb{R}^n : \langle y, v \rangle \leq 0 \ \forall v \in T_C(x) \}.$$

The sets $T_C(x)$ and $N_C(x)$ are closed convex cones in \mathbb{R}^n and $T_C(x) \cap N_C(x) = \{0\}$. In addition, since C is convex, $N_C(x)$ coincides with the cone of normals to C at x in the sense of convex analysis, namely

$$(2.6) \quad N_C(x) = \{ \xi \in \mathbb{R}^n : \langle \xi, x - y \rangle \geq 0 \ \forall y \in C \}$$

(see [Cl, proposition 2.4.4]).

Lemma 1. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$ satisfy (V_1) and (V_2) . Then $\partial V(x) \subset N_{\{V \leq V(x)\}}(x)$ for every $x \in \{V > a\}$.*

Proof. Let $C = \{V \leq V(x)\}$. By (2.1), since C is a closed convex subset of \mathbb{R}^n , it is enough to prove that $\{ \lim \nabla V(x_i) : (x_i) \subset D, \lim x_i = x, \exists \lim \nabla V(x_i) \} \subset N_C(x)$.

Arguing by contradiction, let $(x_i) \subset D$ be such that $\lim x_i = x$ and $\lim \nabla V(x_i) = \xi$ with $\xi \notin N_C(x)$. Then, by (2.6), there exists $y \in C$ such that $\langle \xi, y - x \rangle > 0$. Since $x, y \in C$ and C is convex, $d_C(x + t(y - x)) = 0$ for all $0 \leq t \leq 1$ and so $d'_C(x; y - x) = 0$, that is $y - x \in T_C(x)$.

Hence, setting $v = y - x$ we have

$$(2.7) \quad v \in T_C(x);$$

$$(2.8) \quad x + v \in C;$$

$$(2.9) \quad \langle \xi, v \rangle > 0.$$

Now we build a sequence $(v_i) \subset \mathbb{R}^n$ in the following way: setting $C_i = \{V \leq V(x_i)\}$, we define $v_i = v$ if $v \in T_{C_i}(x_i)$. Otherwise we take $v_i \in \mathbb{R}^n$ such that $|v - v_i| \leq d_{C_i}(x_i + v) + \frac{1}{i}$ and $x_i + v_i \in C_i$. In any case, by the convexity of C_i , we have that

$$(2.10) \quad v_i \in T_{C_i}(x_i).$$

Noting that $d_{C_i}(x_i + v) \leq d_{C_i}(x + v) + |x_i - x|$, since $\lim x_i = x$, we can conclude that

$$(2.11) \quad \lim v_i = v$$

provided that we prove that $\lim d_{C_i}(x + v) = 0$. But this is true, because otherwise there is $\epsilon > 0$ and a sequence $(i_k) \subset \mathbb{N}$ such that $B_{2\epsilon}(x + v) \cap C_{i_k} = \emptyset$ for every $k \in \mathbb{N}$. Without loss of generality, we can also assume that the sequence $(V(x_{i_k}))$ is monotone, so that $\{V < V(x)\} \subseteq \bigcup_{k \geq 1} \bigcap_{h \geq k} C_{i_h}$. Hence $B_{2\epsilon}(x + v) \cap \{V < V(x)\} = \emptyset$ and consequently $B_\epsilon(x + v) \cap C = \emptyset$ in contradiction with (2.8).

Since $\nabla V(x_i) \in N_{C_i}(x_i)$, by (2.10) and (2.6) we get $\langle \nabla V(x_i), v_i \rangle \leq 0$ for every $i \in \mathbb{N}$. Passing to the limit $i \rightarrow \infty$, by (2.11) we conclude that $\langle \xi, v \rangle \leq 0$ in contradiction with (2.9). \square

Now we discuss some properties about the Cauchy problem (\mathcal{P}_ξ) . First of all we point out that the mapping $x \mapsto \partial V(x)$ is an upper semicontinuous compact convex multifunction defined on \mathbb{R}^n (see [Cl]). This implies that for any $\xi \in \mathbb{R}^n$, problem (\mathcal{P}_ξ) admits at least a solution, namely a function $x(\cdot, \xi) \in AC_{loc}(I, \mathbb{R}^n)$, where I is an interval containing 0, such that $x(0, \xi) = \xi$ and $x'(t, \xi) \in \partial V(x(t, \xi))$ for almost every $t \in I$. Moreover, maximal solutions are defined on \mathbb{R} and the reachable set at time t , denoted by $R(t, \xi)$, is connected (see [AC]). In addition the following properties hold.

Proposition 2. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and $a, b \in \mathbb{R}$, $a < b$, satisfy (V_1) – (V_3) .*

- (i) *If $x(\cdot, \xi) \in AC_{loc}(\mathbb{R}, \mathbb{R}^n)$ is a solution of the problem (\mathcal{P}_ξ) for $\xi \in \{V \geq b\}$, then $x(t, \xi) \in \{V > b\}$ for every $t > 0$, the function $t \mapsto V(x(t, \xi))$ is increasing and $\frac{d}{dt} V(x(t, \xi)) > 0$ for almost every $t \in \mathbb{R}$ such that $x(t, \xi) \in \{V \geq b\}$.*
- (ii) *If $V(\xi_1) = V(\xi_2) \geq b$ and $t_1, t_2 > 0$ are such that $V(x(t_1, \xi_1)) = V(x(t_2, \xi_2))$ then $|x(t_1, \xi_1) - x(t_2, \xi_2)| \geq |\xi_1 - \xi_2|$.*
- (iii) *For any $\xi \in \{V \geq b\}$ there is a unique $t \leq 0$ such that $x(t, \xi) \in \{V = b\}$.*

Proof. (i) Let $x(\cdot, \xi) \in AC_{loc}(\mathbb{R}, \mathbb{R}^n)$ be a solution of (\mathcal{P}_ξ) . The function $t \mapsto V(x(t, \xi))$ is locally Lipschitz continuous and then differentiable almost everywhere on \mathbb{R} . First of all we prove that

$$(2.12) \quad \frac{d}{dt}V(x(t, \xi)) \geq 0 \text{ for almost every } t \in \mathbb{R} \text{ such that } x(t, \xi) \in \{V \geq b\}.$$

Let $t \in \mathbb{R}$ be such that $x(t, \xi) \in \{V \geq b\}$, $x(\cdot, \xi)$ and $V(x(\cdot, \xi))$ are differentiable at t and $\frac{d}{dt}V(x(t, \xi)) < 0$. Then there is $\delta > 0$ such that $V(x(t+h, \xi)) < V(x(t, \xi))$ for $0 < h < \delta$. Hence, setting $C_t = \{V \leq V(x(t, \xi))\}$ we have $x'(t, \xi) \in T_{C_t}(x(t, \xi))$ because the tangent cone is closed. On the other hand, by Lemma 1, $x'(t, \xi) \in N_{C_t}(x(t, \xi))$ and this implies that $x'(t, \xi) = 0$, in contradiction with (V_3) . Therefore (2.12) is proved. Now we check that the function $t \mapsto V(x(t, \xi))$ is strictly increasing. By the contrary, let us suppose that there exists an interval $[t_1, t_2]$ in which the function is constant, let us say equal to c . For any $t \in [t_1, t_2]$ and h such that $t+h \in [t_1, t_2]$, by a convexity argument we have $x(t+h, \xi) - x(t, \xi) \in T_{C_t}(x(t, \xi))$ and again a contradiction, as before. Now we prove that $x(t, \xi) \in \{V > b\}$ for every $t > 0$. By the upper semicontinuity of $\partial V(\cdot)$, there is $\epsilon > 0$ and $\delta > 0$ such that $|x'(t, \xi)| \geq \delta$ for almost any t for which $V(\xi) - \epsilon < V(x(t, \xi)) \leq V(\xi)$. Using this remark the conclusion follows arguing as above.

(ii) Let $\bar{b} = V(\xi_1) = V(\xi_2)$ and $c = V(x(t_1, \xi_1)) = V(x(t_2, \xi_2))$. We denote by τ_i the inverse function of $V(x(\cdot, \xi_i))$, defined by $V(x(\tau_i(\alpha), \xi_i)) = \alpha$ for $\alpha \in [\bar{b}, c]$. It is an absolutely continuous increasing function and

$$(2.13) \quad \begin{aligned} & \frac{d}{d\alpha} \frac{1}{2} |x(\tau_1(\alpha), \xi_1) - x(\tau_2(\alpha), \xi_2)|^2 \\ &= \langle x(\tau_1(\alpha), \xi_1) - x(\tau_2(\alpha), \xi_2), x'(\tau_1(\alpha), \xi_1) \tau_1'(\alpha) - x'(\tau_2(\alpha), \xi_2) \tau_2'(\alpha) \rangle \\ &= \langle x(\tau_1(\alpha), \xi_1) - x(\tau_2(\alpha), \xi_2), x'(\tau_1(\alpha), \xi_1) \rangle \tau_1'(\alpha) \\ &\quad + \langle x(\tau_2(\alpha), \xi_2) - x(\tau_1(\alpha), \xi_1), x'(\tau_2(\alpha), \xi_2) \rangle \tau_2'(\alpha). \end{aligned}$$

Now we remark that $\tau_i'(\alpha) = (\frac{d}{dt}V(x(\tau_i(\alpha), \xi_i)))^{-1} > 0$ for almost every $\alpha \in [\bar{b}, c]$. In addition, by lemma 1, $x'(\tau_i(\alpha), \xi_i)$ belongs to the normal cone to $\{V \leq V(x(\tau_i(\alpha), \xi_i))\}$ at the point $x(\tau_i(\alpha), \xi_i)$. Then, by (2.6), the last term in (2.13) is non negative and this proves (ii).

(iii) Arguing by contradiction, let $\xi \in \{V > b\}$ be such that $x(t, \xi) \in \{V > b\}$ for all $t \leq 0$. Then, by the part (i), the function $t \mapsto V(x(t, \xi))$ is strictly increasing on \mathbb{R} and there exists $\lim_{t \rightarrow -\infty} V(x(t, \xi)) = l \in [b, V(\xi))$. Firstly we prove that

$$(2.14) \quad \text{the set } \{x(t, \xi) : t \leq 0\} \text{ is bounded.}$$

Indeed, let us take $\bar{\xi} \in \{V = l\}$. By the part (i), there exists $\bar{s} > 0$ such that $l < V(x(\bar{s}, \bar{\xi})) \leq V(\xi)$. In addition there exists $\bar{t} \leq 0$ such that $V(x(\bar{t}, \xi)) = V(x(\bar{s}, \bar{\xi}))$. Moreover, since $V(x(t, \xi))$ decreases to l as $t \rightarrow -\infty$, for any $t \leq \bar{t}$ there is $s \in [0, \bar{s}]$ such that $V(x(t, \xi)) = V(x(s, \bar{\xi}))$. Hence, by the part (ii), we get $|x(t, \xi) - x(s, \bar{\xi})| \leq |x(\bar{t}, \xi) - x(\bar{s}, \bar{\xi})|$. Setting $r = |x(\bar{t}, \xi) - x(\bar{s}, \bar{\xi})|$ and $K = \{y \in \mathbb{R}^n : |y - x(s, \bar{\xi})| \leq r, s \in [0, \bar{s}]\}$, it holds that K is compact and $x(t, \xi) \in K$ for all $t \leq \bar{t}$. Thus (2.14) is proved.

By (2.14), we can take a sequence $(t_i) \subset \mathbb{R}$ such that $t_i \rightarrow -\infty$, $x(t_i, \xi) \rightarrow \bar{\xi}$ and $x(t_i + 1, \xi) \rightarrow \zeta$ for some $\bar{\xi}, \zeta \in \{V = l\}$. Then, setting $\xi_i = x(t_i, \xi)$, it holds that $x(t, \xi_i) = x(t_i + t, \xi)$ for every $t \in [0, 1]$ and, by (2.14), $\|x'(\cdot, \xi_i)\|_{L^\infty(0,1)} \leq \text{const.}$ Therefore, up to a subsequence, $x(\cdot, \xi_i)$ converges uniformly on $[0, 1]$ to some $y(\cdot) \in AC([0, 1], \mathbb{R}^n)$ and $x'(\cdot, \xi_i) \rightarrow y'(\cdot)$ in the weak* topology of $L^\infty(0, 1)$. By [AC, Theorem 1, pag. 104] the function $y(\cdot)$ is a solution to the problem $(\mathcal{P}_{\bar{\xi}})$ on the interval $[0, 1]$ and $y(1) = x(1, \bar{\xi}) = \zeta$. In addition $l \leq V(y(t)) = \lim V(x(t + t_i, \bar{\xi})) \leq \lim V(x(t_i + 1, \bar{\xi})) = l$, that is $x(t, \bar{\xi}) \in \{V = l\}$ for all $t \in [0, 1]$, which contradicts the part (i). Thus we proved that for any $\xi \in \{V \geq b\}$ there exists at least a value $t \leq 0$ such that $x(t, \xi) \in \{V = b\}$. The uniqueness of such a value t follows from the part (i). \square

We now introduce the notion of geometrically distinct solutions for the Cauchy problem (\mathcal{P}_ξ) .

Definition. Given $\xi \in \mathbb{R}^n$ two solutions $x_i(\cdot, \xi) \in AC_{loc}(\mathbb{R}, \mathbb{R}^n)$, $i = 1, 2$ to the problem (\mathcal{P}_ξ) are geometrically distinct in the future (respectively in the past) if the sets $\{x_i(t, \xi) : t \geq 0\}$ (resp. $\{x_i(t, \xi) : t \leq 0\}$) are different. We say that the problem (\mathcal{P}_ξ) has uniqueness of solution in the future (resp. in the past) if there do not exist two solutions geometrically distinct in the future (resp. in the past).

Remark 1. Property (ii) of Proposition 2 implies that, for any $\xi \in \mathbb{R}^n$, problem (\mathcal{P}_ξ) has uniqueness of solution in the past.

Remark 2. For any $c \geq b$ we can define a function $f_c : \{V = c\} \rightarrow \{V = b\}$ in the following way: $f_c(y) = \xi$, where ξ is such that there exist a solution $x(\cdot, \xi)$ of problem (\mathcal{P}_ξ) and a value $t \geq 0$ satisfying $x(t, \xi) = y$. Property (iii) of Proposition 2 guarantees that this function is well defined and, thanks to (ii), it satisfies the global Lipschitz condition $|f_c(x) - f_c(y)| \leq |x - y|$ for any $x, y \in \{V = c\}$.

Example. When hypothesis (V_3) is not satisfied it may happen that some level set $\{V = c\}$ has Hausdorff dimension greater than $n - 1$. Even assuming that any level set $\{V = c\}$ has Hausdorff dimension $n - 1$, the hypothesis (V_3) plays a fundamental role. Indeed, let us consider the function $V : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$V(x) = \begin{cases} |x| & \text{if } |x| \leq 1 \\ \frac{2}{3}\sqrt{(x-1)^3} & \text{if } x > 1 \\ \frac{2}{3}\sqrt{(-x-1)^3} & \text{if } x < -1. \end{cases}$$

It is easy to check that $0 \in \partial V(x)$ if $x \in \{V = 1\}$. For any $0 < \xi \leq 1$ we have that

$$x_1(t, \xi) = \begin{cases} \xi + t & \text{if } 0 \leq t \leq 1 - \xi \\ 1 & \text{if } t > 1 - \xi \end{cases}$$

and

$$x_2(t, \xi) = \begin{cases} \xi + t & \text{if } 0 \leq t \leq 1 - \xi \\ \frac{1}{4}(t-1)^2 + 1 & \text{if } t > 1 - \xi \end{cases}$$

are two geometrically distinct solutions of the problem (\mathcal{P}_ξ) .

3. PROOF OF THE THEOREM

Let Ω be the set of initial data such that the Cauchy problem (\mathcal{P}_ξ) has at least two solutions geometrically distinct in the future. We denote by \mathcal{H}^s the s -dimensional Hausdorff measure and by \mathcal{L}^n the Lebesgue measure in \mathbb{R}^n .

We want to prove that $\mathcal{H}^{n-1}(\Omega \cap \{V = b\}) = 0$. For $\xi \in \{V = b\}$ and $c \geq b$ we denote $R_c(\xi) = \{y \in \{V = c\} : f_c(y) = \xi\}$ where f_c is given as in Remark 2. We devide the proof in the following steps.

Step 1. For every $\xi \in \Omega \cap \{V = b\}$ there is $c > b$, $c \in \mathbb{Q}$ such that $\mathcal{H}^0(R_c(\xi)) = \infty$.

Let (C_j) be a countable collection of compact sets that cover \mathbb{R}^n .

Step 2. For every $\xi \in \Omega \cap \{V = b\}$ there exists $c > b$, $c \in \mathbb{Q}$ and $\bar{j} \in \mathbb{N}$ such that $\mathcal{H}^0(R_c(\xi) \cap C_{\bar{j}}) = \infty$.

Step 3. Application of the coarea formula.

Step 1. Let $\xi \in \{V = b\}$ be such that there exist two solutions $x_i(\cdot, \xi)$ ($i = 1, 2$) of the problem (\mathcal{P}_ξ) geometrically distinct in the future. We can say that there are a rational $c > b$ and $t_1, t_2 > 0$ such that $x_i(t_i, \xi) \in \{V = c\}$ ($i = 1, 2$) and $x_1(t_1, \xi) \neq x_2(t_2, \xi)$. We define

$$\gamma_i = \{x_i(t, \xi) : t \geq 0\},$$

$$\bar{t}_2 = \sup\{t \in [0, t_2] : d(x_2(t, \xi), \gamma_1) = 0\}$$

$$\xi_1 = x_2(\bar{t}_2, \xi).$$

We remark that $\xi_1 \in \gamma_1 \cap \gamma_2$, there exists \bar{t}_1 such that $\xi_1 = x_1(\bar{t}_1, \xi)$ and $x_i(t - \bar{t}_i, \xi)$ are solutions of problem (\mathcal{P}_{ξ_1}) . Moreover, if $y(\cdot, \xi_1)$ is a solution of (\mathcal{P}_{ξ_1}) in $[0, +\infty)$, then

$$x(t) = \begin{cases} x_i(t, \xi) & \text{for } t \in [0, \bar{t}_i] \\ y(t - \bar{t}_i, \xi_1) & \text{for } t > \bar{t}_i \end{cases}$$

is a solution of (\mathcal{P}_ξ) . Then $R(t, \xi_1) \subset R(t + \bar{t}_i, \xi)$ for any $t > 0$ and, by Proposition 2, part (i), $R(t, \xi_1) \subset \{V > V(\xi_1)\}$ for any $t > 0$. We have also $R(t, \xi_1) \cap \gamma_i \neq \emptyset$ and $R(t, \xi_1) \cap \gamma_1 \cap \gamma_2 = \emptyset$. We can suppose that $R(t, \xi_1) \subset \{V(\xi_1) < V \leq c\}$ if t is sufficiently small. In fact, if it is not the case we can substitute $x_2(\cdot, \xi)$ with an other solution, denoted again by $x_2(\cdot, \xi)$, in such a way the above inclusion holds. We state the following

Claim: there is a solution $x_3(\cdot, \xi)$ of the problem (\mathcal{P}_ξ) and a value $t_3 > 0$ such that

$$x_3(t_3, \xi) \in \{V = c\} \text{ and } x_3(t_3, \xi) \neq x_i(t_i, \xi) \text{ for } i = 1, 2.$$

Indeed for any $\alpha \in \{V(y) : y \in R(t, \xi_1)\}$ and $i = 1, 2$ there is a unique $z_\alpha^i \in \gamma_i \cap \{V = \alpha\}$ and $z_\alpha^1 \neq z_\alpha^2$. Moreover there exists $\alpha \in \{V(y) : y \in R(t, \xi_1)\}$ for which the set $\{V = \alpha\} \cap R(t, \xi_1)$ contains a third point $z_\alpha^3 \neq z_\alpha^i$, $i = 1, 2$. Let $y_3(\cdot, \xi)$ be a solution to the problem (\mathcal{P}_ξ) such that, for a suitable t , it is $y_3(t, \xi) = z_\alpha^3$.

If $\lim_{t \rightarrow +\infty} V(y_3(t, \xi)) > c$ then there exists $t_3 > 0$ such that $V(y_3(t_3, \xi)) = c$ and $y_3(t_3, \xi) \neq x_i(t_i, \xi)$ for $i = 1, 2$. In this case we put $x_3(\cdot, \xi) = y_3(\cdot, \xi)$.

If $\lim_{t \rightarrow +\infty} V(y_3(t, \xi)) \leq c$, by Proposition 2, part (i) $V(y_3(t, \xi)) < c$ for any $t > 0$. Considering the problem $(\mathcal{P}_{x_1(t_1, \xi)})$, we have that for any $t > 0$ $\{V > c\} \cap R(t+t_1, \xi) \supset R(t, x_1(t_1, \xi)) \neq \emptyset$. Moreover $R(t+t_1, \xi) \cap \{V < c\} \neq \emptyset$. Then for any $t > 0$ there exists $z \in R(t+t_1, \xi) \cap \{V = c\}$. We want to prove that $z \neq x_i(t_i, \xi)$. Let us suppose that $z = x_1(t_1, \xi)$. Then $x_1(t_1, \xi) \in R(t+t_1, \xi)$ for any $t > 0$. In other words, for any $t > 0$ we can find a solution $y(\cdot)$ to the problem (\mathcal{P}_ξ) such that $y(t+t_1) = x_1(t_1, \xi)$ and for any $\tau \in [0, t+t_1]$ there exists $s \in [0, t_1]$ such that $y(\tau) = x_1(s, \xi)$. For $s \in [0, t_1]$ the curve described by $x_1(s, \xi)$ (the same described by $y(\tau)$ for $\tau \in [0, t+t_1]$) is contained in a compact set $K \subset \{b \leq V \leq c\}$, so there exist two positive constants m, M such that for any $v \in \partial V(x)$, $x \in K$ it is $m \leq |v| \leq M$. Computing the length of this curve we have

$$m(t+t_1) \leq \int_0^{t+t_1} |y'(\tau)| d\tau = \int_0^{t_1} |x_1'(s, \xi)| ds \leq Mt_1$$

and we get that $t \leq t_1 (\frac{M}{m} - 1)$, contradicting the fact that t is an arbitrary positive value. Then repeating the same argument for $x_2(t_2, \xi)$, and choosing t sufficiently large, we can denote by $x_3(\cdot, \xi)$ a solution to the problem (\mathcal{P}_ξ) for which there exists t_3 such that $x_3(t_3, \xi) = z$. Hence the claim is proved.

Let now $\gamma_3 = \{x_3(t, \xi) : t \geq 0\}$. We can suppose without restriction that $\gamma_1 \cap \gamma_3 \subset \gamma_2 \cap \gamma_3$. Let $\bar{t}_3 = \sup\{t \in [0, t_3] : d(x_3(t, \xi), \gamma_2) = 0\}$ and $\xi_2 = x_3(\bar{t}_3, \xi)$. Now, we can repeat the same argument to construct a sequence $(x_i(\cdot, \xi))$ of solutions of the problem (\mathcal{P}_ξ) and a corresponding sequence $(t_i) \subset \mathbb{R}^+$ such that $x_i(t_i, \xi) \in \{V = c\}$ and $x_i(t_i, \xi) \neq x_j(t_j, \xi)$ for $i \neq j$. Hence we have that $x_i(t_i, \xi) \in R_c(\xi)$ for any $i \in \mathbb{N}$ and then $\mathcal{H}^0(R_c(\xi)) = \infty$, being \mathcal{H}^0 the counting measure.

Step 2. Arguing by contradiction, let us suppose that for any $\epsilon > 0$ and $k > 0$ there exists $i_{\epsilon, k} \in \mathbb{N}$ such that for any $i > i_{\epsilon, k}$ it is $|x_1(t_1(b+\epsilon), \xi) - x_i(t_i(b+\epsilon), \xi)| > k$. Hence $\lim_{\epsilon \rightarrow 0+} |x_1(t_1(b+\epsilon), \xi) - x_i(t_i(b+\epsilon), \xi)| \geq k$ and this contradicts the fact that for any $i \in \mathbb{N}$ $\lim_{\epsilon \rightarrow 0+} x_i(t_i(b+\epsilon)) = \xi$.

Step 3. By the step 2, we have

$$\{V = b\} \cap \Omega = \bigcup_{j \in \mathbb{N}} \bigcup_{c \in \mathbb{Q}, c > b} \{\xi : \mathcal{H}^0(R_c(\xi) \cap C_j) = \infty\}.$$

We now apply the coarea formula (see [M, Theorem 3.13])

$$\int_{\{V=c\} \cap C_j} Jf_c(x) d\mathcal{H}^{n-1}(x) = \int_{\{V=b\}} \mathcal{H}^0(R_c(\xi) \cap C_j) d\mathcal{H}^{n-1}(\xi).$$

The left hand side is finite because f_c is Lipschitzean and $\mathcal{H}^{n-1}(\{V = c\} \cap C_j) < \infty$. Then for any $c \in \mathbb{Q}$, $c > b$ and for any $j \in \mathbb{N}$

$$\mathcal{H}^{n-1}(\{V = b\} \cap \{\xi : \mathcal{H}^0(R_c(\xi) \cap C_j) = \infty\}) = 0.$$

□

We conclude this section proving the Corollary.

Proof of the Corollary. We can apply the Theorem with $a = \inf V$. Then $\mathcal{H}^{n-1}(\{V = b\} \cap \Omega) = 0$ for any $b > a$ and using again the coarea formula, we get

$$\int_{\Omega \cap C_j} JV(x) d\mathcal{H}^n(x) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\{V = b\} \cap \Omega \cap C_j) d\mathcal{H}^1(b) = 0$$

where (C_j) is a covering of \mathbb{R}^n of compact sets. By (V_3) , $JV(x) > 0$ for \mathcal{H}^n -almost every $x \in \mathbb{R}^n \setminus \{V = a\}$. Moreover, if $\{V = a\} \neq \emptyset$, the set $\Omega \cap \{V = a\}$ is contained in the boundary of the convex set $\{V = a\}$ in \mathbb{R}^n , denoted $\partial\{V = a\}$. Indeed if $\xi \in \mathbb{R}^n$ is an interior point of $\{V = a\}$ then the only solution to (\mathcal{P}_ξ) is the constant function $x(t, \xi) = \xi$. Since the Hausdorff dimension of $\partial\{V = a\}$ is strictly less than n , $JV(x) > 0$ for \mathcal{H}^n -almost every $x \in \Omega$. Therefore $\mathcal{H}^n(\Omega \cap C_j) = 0$ for any $j \in \mathbb{N}$. Since \mathcal{H}^n coincides with the Lebesgue measure \mathcal{L}^n , we have that Ω is Lebesgue measurable and $\mathcal{L}^n(\Omega) = 0$. □

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